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NON-METRIC GRAVITY: A STATUS REPORT

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We review the status of a certain (infinite) class of four-dimensional generally covariant theories propagating two degrees of freedom that are formulated without any direct mention of the metric. General relativity itself (in its Plebański formulation) belongs to the class, so these theories are examples of modified gravity. We summarize the current understanding of the nature of the modification, of the renormalizability properties of these theories, of their coupling to matter fields, and describe some of their physical properties.

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1. Introduction

General relativity (GR) was formulated by Einstein as a theory that dynamically determines the spacetime metric. This theory is famously non-renormalizable, which means that it gives only the “low-energy” description of gravity. Because the Newton’s constant is so small, what is low energy for gravity may be very high energy by the standards of particle physics, so for all practical purposes we are quite happy with the description of gravity given by GR. The problem of quantum gravity, which is to find a satisfactory description of gravity at “high” (Planckian) energy is then a very hard one as it is not possible, and is unlikely to be possible, to probe the relevant range of energies by direct experiments. In spite of our almost absolute ignorance as to what happens at Planckian energy, the common consensus is that it is unlikely that the metric description of gravity survives there. For this reason many alternative descriptions have been developed, some of them rather radical, e.g., even calling for abandoning of the notion of the spacetime manifold. The most popular approach to the problem is given by string theory, which is well defined as a quantum theory at the expense of introducing extra symmetries and dimensions that so far are not observed.

It is not universally appreciated, however, that general relativity itself admits a rather radical reformulation that almost entirely eliminates the spacetime metric

from the picture. This formulation was discovered a long ago by Plebański¹, and was rediscovered more recently in² after the work on the new Hamiltonian formulation for gravity by Ashtekar³. Ashtekar's new formulation of GR can be seen² to be just the Hamiltonian formulation of the Plebański theory. The Plebanski formulation, together with the idea of describing gravity by something else than the spacetime metric, leads to a class of gravity theories much larger than GR, which is the subject of this review.

In Plebański formulation of gravity, the notion of the dynamical spacetime metric is replaced by that of the dynamical Hodge operator. Recall that, given a metric $g_{\mu\nu}$, the Hodge operator can be defined as the map acting on the space of two-forms and mapping a two-form $B_{\mu\nu}$ into its “dual” two-form $(*B)_{\mu\nu} = (1/2)\epsilon_{\mu\nu}^{\rho\sigma}B_{\rho\sigma}$. Here $\epsilon_{\mu\nu\rho\sigma}$ is the volume four-form compatible with the metric. It is easy to check that the Hodge dual operator is invariant under conformal rescalings of the metric $g_{\mu\nu} \rightarrow \Omega g_{\mu\nu}$. The key fact is that the converse is also true and two metrics that define the same Hodge operator are related by a conformal transformation, see, e.g.,⁴ for a simple proof. This means that the Hodge dual operator defines a metric up to conformal transformations, and that it can be used as the main dynamical object in a theory of gravity, instead of the metric.

This is essentially the idea that was realized in¹ (even though it is only implicit in this work). More precisely, to describe the Hodge operator it is sufficient to specify which two-forms are self-dual. The self-dual forms are those satisfying $*B = \sqrt{\sigma}B$, where $\sigma = *^2$, so $\sigma = \pm 1$ for the Euclidean and Lorentzian signatures, respectively. Knowing the subspace W^+ of self-dual two-forms, the anti-self-dual forms $*B = -\sqrt{\sigma}B$ are found as those orthogonal to self-dual ones with respect to the scalar product in the space of two-forms given by the wedge product (the scalar product itself depends on the choice of a volume form; the notion of the orthogonality, however, does not). The knowledge of the subspaces W^\pm is equivalent to the knowledge of the Hodge operator, as is not hard to see.

The description then proceeds as follows. One introduces three two-form fields B^i . Declaring the subspace (in the space of two-forms) spanned by the fields B^i to be that of self-dual two-forms determines the metric up to a conformal factor. This last is fixed by a choice of the volume form, for which there is a natural choice $(vol) = (1/3)\delta^{ij}B^i \wedge B^j$, where δ_{ij} is the Kronecker delta. The metric defined by this data is then invariant under $SO(3)$ rotations of the triple B^i . This suggests that we should think of the triple B^i as of a section of a vector bundle $V \rightarrow M$ associated with a principal $SO(3)$ bundle over the spacetime manifold M . Therefore, in addition to B^i , we should introduce a connection A^i on V . It is now not hard to write the field equations for the theory. First, it is natural to require the connection A^i to be compatible with the triple B^i , i.e., require $(D_A B)^i = 0$. Given a triple B^i , this set of 4×3 equations can be solved for 4×3 components of A^i in terms of derivatives of B^i . It remains to write an equation that would allow one to find B^i . It is natural to require that this equation be second order in derivatives and

SO(3) covariant. This points to some equation that involves $F^i(A)$, the curvature of the connection A^i . The obvious choice $F^i(A) = 0$ does give 6×3 equations for 6×3 components of B^i . However, the theory that one gets this way is not very interesting as it is void of any physics (does not have local degrees of freedom). A much more interesting choice is:

$$F^i(A) = \Lambda_j^i B^j, \quad (1)$$

where Λ_j^i is some (undetermined) matrix. This equation says that the curvature is purely self-dual as a two-form. This gives us 3×3 equations $(F^i(A))_{asd} = 0$ for 6×3 components of B^i so the theory is not well-defined as we cannot solve for the two-form fields. One can make it into a well-defined theory by writing an action principle that leads to (1), namely:

$$S[B, A, \Lambda] = \int B_i \wedge F^i(A) - \frac{1}{2} \Lambda_{ij} B^i \wedge B^j, \quad (2)$$

where we raise and lower indices using δ^{ij} . Varying this action with respect to A^i we get the equation $(D_A B)^i = 0$, while varying it with respect to B^i gives (1). Note that only the symmetric part of Λ_{ij} enters equation (2), so this matrix should be assumed symmetric also in (1). The action (2) tells us that we should treat Λ_{ij} as a dynamical object and vary with respect to it. We get: $B^i \wedge B^j = 0$, which gives six equations for B^i . Together with (1) this gives 15 equations. Noting that three components of B^i are pure gauge we thus get the right number of equations to determine B^i completely. The components of the symmetric matrix Λ_{ij} are determined from the unused equations in (1). Note, however, that the equation $B^i \wedge B^j = 0$ implies, in particular, that the volume form (*vol*) constructed above is zero. Also, it is rather obvious that there is no Hodge dual operator for which the space W^+ of self-dual two-forms is null, $W^+ \wedge W^+ = 0$. So, this theory is not of any physical interest.^a

To get an interesting theory, we will impose one constraint on the symmetric matrix Λ_{ij} that appears on the right-hand-side of (1). If we do so, then only five of the six components of Λ_{ij} remain independent. Equation (1) then gives ten equations for the components of B^i (this is the number of equations in GR) plus five equations for the independent components of Λ_{ij} in terms of the second derivatives of B^i . Variation of the action (2) with respect to the independent components of Λ_{ij} gives an additional five equations on B^i , which gives just enough equations to determine the two-form fields.

What constraint can be imposed on Λ_{ij} ? It is clear that the five independent components of Λ_{ij} can only be chosen to be those of its *traceless* part, which we denote by $\Lambda_{ij} - (1/3)\delta_{ij}\text{Tr}(\Lambda) := \Psi_{ij}$. Indeed, this is the part that forms an irreducible representation with respect to the action of SO(3). The trace part $\text{Tr}(\Lambda)$ then becomes a function of its traceless part Ψ_{ij} that we denote by $-\phi(\Psi)$ (the

^aIts Hamiltonian analysis shows that this is simply GR with the Hamiltonian constraint removed.

minus sign is to agree with certain conventions; see below). Thus, we have:

$$\Lambda_{ij} = \Psi_{ij} - \frac{1}{3}\delta_{ij}\phi(\Psi). \quad (3)$$

The theory defined by the action (2) where only the traceless part Ψ_{ij} of Λ_{ij} is an independent variable (3) gives the right number of equations to solve for the components of the two-form field B^i and thus determine the Hodge dual operator, which in turn (together with the volume form) determines the metric. We also note that the volume form $(1/3)\delta_{ij}B^i \wedge B^j$ is no longer trivial: the equation one obtains by varying the action (2) with respect to Ψ_{ij} ,

$$B^i \wedge B^j - \frac{1}{3}\delta^{ij}(B^k \wedge B_k) = \frac{1}{3}\frac{\partial\phi}{\partial\Psi_{ij}}(B^k \wedge B_k), \quad (4)$$

no longer implies any relation for the volume form. We thus get a theory with potentially interesting physical implications. It becomes even more interesting after one observes that, for $\phi(\Psi) = \Lambda = \text{const}$, the above theory is nothing else but the Einstein's GR in disguise (i.e., in its Plebański formulation), with Λ being the cosmological constant (our choice of the sign in (3) was motivated precisely by the desire to agree with the usual convention for the sign of Λ). Therefore, what we have obtained is an infinite family of generalizations of general relativity, where to fix a theory one has to specify the function $\phi(\Psi) = \phi(\text{Tr}(\Psi^2), \text{Tr}(\Psi^3))$ of two invariants that can be constructed from the traceless matrix Ψ . The above counting of the field equations suggests that this theory makes sense for an arbitrary $\phi(\Psi)$. A more detailed analysis, whose results we will review in the next section, shows that this is indeed the case.

Our above discussion was motivated by the idea to describe gravity as a dynamical theory of an object other than the spacetime metric. We have seen how GR itself can be reformulated in these terms, and how an infinite class of gravity theories different from GR and parametrized by a function $\phi(\Psi)$ is obtained this way. In the next section we will review the (classical) properties of this class of theories without worrying too much about the principle that can fix ϕ . We will see that, for an arbitrary ϕ , the gravity theory (2) resembles GR in many ways. Still, there are some new physical effects predicted by this theory. It is important to emphasize that, at the level of the classical physics, nothing forces us to depart from the familiar ground of GR, and the new class of theories described above may be viewed as not more than a mathematical curiosity. However, when one considers the quantum theory of usual GR in Plebański (Hodge operator) formulation, one *is* forced to consider theories more general than GR, similar to what happens in quantum gravity in the metric formulation. In the usual metric formulation of quantum gravity, one has to introduce theories with higher derivatives and, thus, with rather unpleasant properties (instability etc.). In the Hodge operator formulation of gravity, many (if not all; see Section 3) terms that have to be added to the action do not change the character of the theory so drastically, i.e., do not introduce higher derivatives and new DOF. The quantum corrected theory is one of the class (2)

with (3). Thus, one does have to take this class of gravity theories seriously, as the modifications they describe will be induced by quantum corrections. It is a different question whether any of these quantum corrections can survive and be of relevance at low (astrophysical) energy scales. It can only be answered after the quantum mechanical behavior of this class of theories is understood. But even in the absence of such understanding, it is interesting to quest what kind of new physical effects can be expected.

The theory (2) with (3) was first proposed in ⁵, where renormalizability properties of GR in Plebański formulation were considered. It was only later appreciated that it is also quite interesting as a purely classical theory, the point of view which was developed in ^{6, 7}. The Hamiltonian analysis of this theory (in a slightly different version) was done in ⁸ and more recently in ⁹. We will describe the main results of the classical analysis in the next section. The current understanding of the quantum mechanical behavior of this class of theories is reviewed in Section 3.

2. Classical properties

2.1. Hamiltonian formulation

The canonical description of the class of theories introduced above is easily obtained. This was done in the “pure connection” formulation in ⁸ and starting directly from (2) in a more recent work ⁹. One finds, exactly like in the usual GR–Plebański–Ashtekar case, that the phase space is parametrized by the canonically conjugate pairs $(A_a^i, \tilde{\sigma}^{ai})$, where A_a^i is the spatial component of the connection, and $\tilde{\sigma}^{ai}$ is the momentum (tilde denotes the density weight). The theory is fully constrained (no Hamiltonian). The constraints are: the Gauss constraint $D_a \tilde{\sigma}^{ai} = 0$, where D_a is the spatial covariant derivative with respect to A_a^i ; the diffeomorphism constraint $\tilde{\sigma}^{ai} F_{ab}^i = 0$, which both take exactly the same form as they do in Plebański theory (or the new Hamiltonian formulation ³); and, finally, the Hamiltonian constraint that becomes:

$$\epsilon^{ijk} F_{ab}^i \tilde{\sigma}^{aj} \tilde{\sigma}^{bk} + \frac{1}{3} \epsilon_{abc} \epsilon^{ijk} \tilde{\sigma}^{ai} \tilde{\sigma}^{bj} \tilde{\sigma}^{ck} \phi \left((F_{ab}^{(i} \epsilon^{j)kl} \sigma^{ak} \sigma^{bl})_{\text{tr-free}} \right) = 0, \quad (5)$$

where the trace-free part of the matrix in the argument of the function ϕ is taken. Note that the momentum variables used in the argument of ϕ have no density weight. For $\phi = \Lambda = \text{const}$, the above Hamiltonian constraint is just that of the Ashtekar Hamiltonian formulation ³ of GR with the cosmological constant.

The constraints described are first class, see ⁹ for a verification of this. Counting of the degrees of freedom (DOF) is then exactly the same as in GR: we have 3×3 configurational DOF, minus three Gauss constraints, minus three diffeomorphism constraints, minus one Hamiltonian constraint, which gives two propagating degrees of freedom. The class of theories in question is thus an infinite (parametrized by a function of two variables) class of four-dimensional generally covariant generalizations of GR propagating two degrees of freedom.

2.2. “Curvature” dependent cosmological “constant”

We have just seen that, at the level of the canonical formulation, the only modification as compared with the usual GR case is that the cosmological constant gets replaced by a non-trivial (and arbitrary) function ϕ of the “curvature” $(F_{ab}^{(i)j)kl}\sigma^{ak}\sigma^{bl})_{\text{tr-free}}$. In GR–Plebański theory this symmetric traceless tensor is nothing else but the Weyl part of the Riemann curvature tensor. For this reason, we will continue to refer to this tensor as “curvature” for any theory of the class under consideration. Note that this tensor has dimension of $1/L^2$, L being length, as is appropriate for the curvature.

The main physical implication of this modification is that the cosmological constant observed in the regions of relatively low curvature does not have to be the same as that observed in the regions of relatively high curvature. In particular, the cosmological constant that appears in the Friedmann equations governing the evolution of a homogeneous isotropic Universe is given by $\lambda_{\text{cosmological}} = \phi(0)$ (the argument of the function ϕ in this case is zero due to symmetries) and does not have to be equal to the large curvature effective cosmological constant at Planck scales. This gives a possible mechanism for solving the “cosmological-constant problem”, which is to explain why the observed cosmological constant is so different from the cosmological constant of the order $1/l_p^2$, where l_p is the Planck length, expected to be induced by the Planck-scale physics. The cosmological constant induced at Planck scales $\phi(1/l_p^2)$ (in the regime of extremely high curvatures $1/l_p^2$) may well be of the order of $1/l_p^2$, but this would not have any observable effect provided the value $\phi(0)$ of the function ϕ at zero curvature is small. The challenge is then to explain what type of physics fixes the form of the function ϕ , and show that the physical ϕ indeed has the properties required. We will return to this (open) question in the next section.

2.3. Homogeneous isotropic cosmology

As we have already mentioned, in the case studied by Friedmann, due to high symmetry, the “curvature” tensor in the argument of the function ϕ vanishes, and predictions of the theory with non-trivial ϕ are the same as those of GR with the cosmological constant $\phi(0)$. Thus, the Friedmann equations do not get modified. However, the theory of cosmological perturbations does get modified. Work is currently in progress to study these modifications.

2.4. New physical effects in the spherically symmetric case

The spherically symmetric problem for the theory in question was solved in ⁷. It was shown that the class of theories under study admits an analog of the Birkhoff’s theorem: a spherically symmetric solution is necessarily static. Importantly, in this case there is just one invariant that can be constructed from the matrix Ψ : $\text{Tr}(\Psi)^2 \sim \beta^2$, $\text{Tr}(\Psi)^3 \sim \beta^3$, so the function ϕ becomes that of a single argument $\phi = \phi(\beta)$.

To describe the main new physical effects that appear in the case of a non-trivial function ϕ , let us assume that this function has the form of a step-function, taking one, approximately constant value in the region of relatively large curvatures and another constant value in the region of relatively small curvatures. Thus, we are envisaging a scenario in which the function ϕ defines a scale, which is the curvature scale at which the change takes effect. In other words, the curvature scale is defined as the value of β for which the dimensionless quantity $\partial\phi/\partial\beta$ is significantly different from zero.

Assuming that the length scale defined by ϕ is sufficiently large (as compared to the horizon radius of a central body), close to the horizon we will have the region of “large” curvatures described by a (dS) Schwarzschild solution. As one goes further away from the horizon, the curvature β starts decreasing as $\beta \sim r_s/r^3$ (r_s is the Schwarzschild radius) and one will eventually enter the region in space (and in the curvature space) where the function ϕ starts to change. One finds that when the modulus of $\partial\phi/\partial\beta$ is nowhere of the order of unity the curvature β (which no longer has the interpretation of the Weyl curvature) continues to decrease, and one eventually enters the other region of constant ϕ . In this region the solution is again (dS) Schwarzschild, but with, in general, a different value of the cosmological constant. The main new physical effects, apart from the changing cosmological constant, are: (i) The observed value of the mass of the spherically symmetric object is different in the “high” and “low” curvature regions; (ii) There is an additional redshift occurring as compared to the case of the usual Schwarzschild. These effects are described by the following simple formulae:

$$\frac{r_s(\beta_1)}{r_s(\beta_2)} = Z(\beta_1, \beta_2), \quad \frac{f(\beta_2)}{f(\beta_1)} = \frac{g(\beta_1)}{g(\beta_2)} Z(\beta_1, \beta_2), \quad (6)$$

where f^2 is the 00 component of the metric, $g^{-2}(r) = 1 - (r_s/r) - (1/3)\phi r^2$ is its inverse rr component, as usual for a (dS) Schwarzschild solution and

$$Z(\beta_1, \beta_2) = e^{\int_{\beta_1}^{\beta_2} \frac{\phi_\beta}{6\beta} d\beta} \quad (7)$$

is the redshift factor. Note that we are using the sign convention for ϕ different from that in ⁷.

To describe the effect of mass “renormalization” in words, let us assume that the function ϕ increases with curvature, which is consistent with the assumption that ϕ is large at Planck scale curvatures and very small at zero curvature. Then if we take for $\beta_{1,2}$, $\beta_1 < \beta_2$ the characteristic values for “small” (far away from the object) and “large” (close to the body) curvatures, respectively, the “redshift” factor is greater than one, which means that the apparent gravitating mass of the object *increases* as one moves further away from it. This simple observation may be able to explain the phenomenon of missing mass (“dark matter”) as a purely gravitational effect. Some order-of-magnitude estimates as to what the relevant curvature scale l must be for such an explanation to be possible are given in ⁷.

The reader should keep in mind that the above discussion, as well as that in ⁷, are just first attempts to extract physical predictions from the new theory, and that a concrete model for “dark matter”, together with a mechanism that would fix the form of the function ϕ , is yet to be proposed. We find it very encouraging, however, that the direction in which the mass of a spherically symmetric object gets “renormalized” (under the assumption of a function ϕ increasing with curvature) is not in conflict with the phenomenon of missing mass.

The other effect that the modified gravity theory predicts is that of an additional redshift (blueshift). The above formula tells us that when light is emitted in the region of “high” curvatures and travels away from the body into the region of “low” curvatures, in addition to the usual relativistic redshift, there will be (under the assumption that ϕ is an increasing function of the curvature) an additional blueshift that will act in the opposite direction, increasing the photon’s energy. If, however, light travels from the regions of “low” curvature to those of “high” curvature close to the body, then there will be an additional redshift effect. As is discussed in ⁷, this effect may be of astrophysical significance.

2.5. Avoidance of the black-hole singularity

Yet an additional reason to take the class of theories described seriously is that the behavior of all fields inside a black hole is much less dramatic than in GR. Let us, as before, assume that the function $\phi(\beta)$ increases as the curvature increases, e.g. becomes of order $1/l_p^2$ at Planckian curvatures. The derivative ϕ_β of ϕ would also normally grow. Then somewhere inside the black hole a point will be reached where ϕ_β becomes $1/3$ (this value is due to the chosen coefficient in front of ϕ in (3)). As a detailed analysis of ⁷ shows, at this point the metric to be constructed from the two-forms B^i becomes no longer defined (singular). However, all the dynamical fields of the theory, namely both B^i and A^i , remain finite. Thus, one can evolve through this surface and enter a new region, where the metric again becomes defined, and which is absent in the Schwarzschild solution. The resulting conformal diagram is given in ⁷. Thus, the theory no longer “carries the seeds of its own destruction”, something of great importance for its logical consistency. Of course, the removal of the singularity in the spherically symmetric solution is not equivalent to the absence of singularities in all possible situations, but this result possibly indicates a much stronger “avoidance of singularity” property of the theory.

2.6. Coupling to matter

Before one can take the theory described seriously one must make sure that matter degrees of freedom can be coupled to it. In general relativity this is straightforward, as matter couples directly to the metric whose dynamics the theory describes. From the construction of our theory as it was presented in the Introduction it was clear that it is the Hodge operator that should be treated as the fundamental object of the theory, not the metric defined by it. It is thus a crucial question if matter fields

can be consistently coupled directly to our basic dynamical fields, which are the triple of two-forms B^i and the connection A^i . For gauge fields, this task is easy, as it is exactly the Hodge operator, not the metric, that is required to write down the Yang-Mills (YM) action. The action that couples the YM gauge field directly to B^i reads ²:

$$S_{YM}[a, \varphi] = \int \varphi_i^\alpha B^i \wedge f^\alpha(a) - \frac{1}{2} \varphi_i^\alpha \varphi_j^\alpha B^i \wedge B^j, \quad (8)$$

where α, β are Lie algebra indices for the gauge group in question, φ_i^α is an auxiliary field, similar to Ψ_{ij} in the case of gravity, and $f^\alpha(a)$ is the curvature of the YM connection a^α . It is not hard to show that, in the geometric optics approximation, this theory describes massless quanta propagating along the null geodesics of the metric defined by B^i . This holds for a general two-form field, independently of whether B^i satisfies Plebański or generalized field equations. Thus, it is encouraging that the gauge fields can be coupled to the class of gravity theories under consideration so seamlessly.

Coupling to other fundamental fields whose existence we know for sure — fermions — is a much more tricky business. The action proposed for this purpose in ² *does not extend* beyond the case of GR. The reason for this is that the field equations that follow from the action in ² form an over-constrained system, and this system is only consistent when the triple B^i satisfies equations (4) with zero right-hand-side, i.e., when the gravity theory is GR. For a theory with general ϕ the combined theory gravity + fermions with the action proposed in ² is simply inconsistent. One therefore has to look for a different description of fermions. Work on this important issue is currently in progress.

3. Renormalization

3.1. Counterterms

Renormalizability properties of general relativity in Plebański formulation were considered in ⁵. Simple power counting arguments show that an infinite set of counterterms must be added to the action, and that in this sense the theory is non-renormalizable, as expected. However, in Plebański formulation, there is more field redefinition freedom than in GR. After using the available field redefinitions one easily shows that there is one infinite set of counterterms that gets combined into the function $\phi(\Psi)$ that we introduced in (3), and another (also infinite) set of counterterms containing covariant derivatives of Ψ_{ij} . These other terms, not considered in the theory (2), are, schematically, of the form $\Psi \dots \Psi (D\Psi)^4$, $\Psi \dots \Psi (D\Psi)^2 B$, $\Psi \dots \Psi (D\Psi)^2 F$, where $\Psi \dots \Psi$ stands for a product of the matrices Ψ_{ij} . With the addition of such terms to the action, the field equations become higher order in derivatives and the (relative) simplicity of the theories (1) is lost. This is reminiscent of what happens in the usual metric-based formulation, where one is forced to add to the action counterterms that lead to higher-derivative field equations. The main difference between the two cases is that, in the Hodge operator based

theory, one can incorporate an infinite number of quantum corrections into the action *without* significantly changing the properties of the theory, while in the metric based gravity even the simplest quantum induced modification has rather dramatic consequences.

It is clear that quantum corrections described by the action (2) can be expressed in the purely metric formulation. Indeed, we have described in the Introduction how the theory (2) describes the metric of the spacetime, albeit implicitly. Thus, for a given ϕ , field equations can be, at least in principle, expressed as field equations for the metric tensor. It is clear that (for a non-constant ϕ) one will get equations involving higher derivatives of the metric. It is also clear that an infinite expansion in derivatives must be present, as no finite number of derivatives would lead to a theory with just two propagating degrees of freedom (higher derivatives lead to higher number of DOF as they require more initial data). Thus, the theory (2) describes an infinitely large class of purely metric theories with infinite number of derivatives, where the sums in derivatives have been re-summed to get a theory with two propagating DOF. Thus, the class of theories under consideration does describe the usual quantum corrected metric gravity. What is unusual is the parametrization of these quantum corrections.

For a more general class of theories, namely those containing terms involving the derivatives of Ψ_{ij} , a relation to the purely metric formulation is not so clear. However, it can be expected that they also correspond to GR plus an infinite set of higher derivative terms, that set being more general than the one that arises from (2).

3.2. *Renormalizability conjecture*

The main open problem is whether the class of quantum corrections described by (2) is complete. In other words, the main question is whether the terms containing the derivatives of Ψ_{ij} that can be added to the action on dimensional grounds are indeed necessary as counterterms if one starts from a theory that does not contain such terms. Our usual experience with quantum fields suggests that they are: what can be added to the action should be added. However, there are some rather strong indications that make the present author to believe that the theories (2) are different in this respect.

First, the author considered a “gauge fixed version” of the Plebański theory with the action given by:

$$S_{g.f.} = \frac{1}{2} \int (\Lambda^{-1})_{ij} F^{+i}(A) \wedge F^{+j}(A), \quad (9)$$

where a background metric is chosen and F^{+i} is the self-dual part of the two-form field strength F^i in this metric, and Λ_{ij} is given by (3). The above expression gives the “bosonic” part of the action, which also contains some ghost and gauge-fixing terms. Apart from the presence in (9) of a non-trivial prefactor $(\Lambda^{-1})_{ij}$ the theory studied by the author is the same as the so-called Donaldson theory; see

^{10,11}. The action (9), together with the other (ghost) terms can be obtained from (2) by a formal gauge fixing procedure. The theory (9) can be shown to behave rather nicely under the renormalization. In particular, for this theory no terms containing derivatives of Λ_{ij} arise as counterterms. This was reported in ⁵, where it was also conjectured that the same property holds for the class of theories (2). The “gauge fixing” procedure that leads from the gravity theory of interest to (9) is only understood by the author at the formal level, this is why a detailed derivation of these results is still unpublished. A completely satisfactory treatment would include answering the question how the graviton degrees of freedom are represented by the gauge fixed action, a question currently beyond the author’s understanding. Thus, the “renormalizability” property of the class of theories (2), namely the property that it is closed under the renormalization, remains a conjecture.

Some further indications in support of the conjecture have started to emerge more recently. The conjecture can only be true if the class of theories (2) possesses some (presumably hidden) symmetry that prevents the terms containing derivatives of Ψ_{ij} from appearing. If this is so, what can this symmetry be? It has recently been realized by the author that one way to view the theories (2) is to regard them as the topological BF theory (whose action is given by the first term in (2)) in which a part of the topological symmetry is *gauge fixed* by the second Λ_{ij} term. Different amount of gauge fixing leads to different theories. Thus, e.g., the theory (2) with all components of Λ_{ij} considered independent removes more gauge symmetries of BF theory than it is necessary to get GR and leads to an uninteresting model. If one accepts this point of view seriously, it then starts to look like the true gravity theory is given by the topological BF theory “gauge fixed” by some mechanism to be understood. We are then seeing gravity simply because we have means of asking the topological BF theory non gauge-invariant questions. This can only be the case if we have fields coupling to our theory in a non-gauge invariant way, which is indeed the case: the coupling of matter fields to the two-form fields B^i breaks the topological symmetry of BF theory. This points in the direction of matter fields as being responsible for the “gauge fixing” that leads to gravity and for an “illusion” of purely gravitational degrees of freedom. Needless to say, these ideas are in their very preliminary stages of development. But they point out in the direction of the topological symmetry of BF theory as being behind the scene in the theories of gravity under consideration. In the opinion of the author, it is a better understanding of all these issues that will one day help to establish the true status of the “renormalizability” conjecture.

3.3. *If the conjecture is true*

This is the most optimistic scenario which does not conflict with anything known. The renormalizability property would give us a window into the Planck scale physics, as we could then conclude that the gravity theory at the Planck scale is a (quantum) theory from the class (2), with some function ϕ that needs to be determined. In

spite of our ignorance as to the form of this function, the above statement would be extremely strong, as it would give us means to make at least qualitative statements about the Planck-scale physics.

What about the form of the function ϕ ? If the conjecture is true, then the renormalization group flow for theories (2) is a flow in the (infinite dimensional) space of functions ϕ . The best-case scenario is then that of the asymptotic safety of Weinberg¹², which is that this renormalization group flow has a non-trivial and reasonably well-behaved ultra-violet fixed point. It would then make sense to use the fixed point theory as the gravitational theory singled out by its extremely appealing renormalizability properties. This is one possible scenario for fixing the form of the function ϕ . Note, however, that the coupling of gravity to matter will most probably have a strong effect on its UV behavior, so it would be unreasonable to study this question in the domain of pure gravity. But the idea of fixing the form of the function ϕ to be that at the UV fixed point can as well be used when matter is present. This gives at least a preliminary scenario. Clearly much more work is required before a “correct” scenario for fixing the form of ϕ is found.

3.4. *If the conjecture is false*

This is what one’s experience with quantum fields in Minkowski spacetime would suggest. If this is the case, one is back where one started: quantum gravity is non-renormalizable, and its quantization does not seem to give any insight on how it behaves at Planckian energies. Nevertheless, the class of theories described in the present review may be of interest even in this case. As we have described above, the theories (2) do incorporate at least some of the quantum corrections to GR. One may then argue that, at “low” energies, the terms containing the derivatives of Ψ_{ij} are less important and that the low energy limit of the quantum corrected theory of gravity is given by (2). This is similar to what we know to happen with the usual metric-based gravity: the terms containing higher derivatives become insignificant at low energies and drop out. It would still be a challenge to explain how a non-trivial function ϕ in (3), if viewed as a result of quantum corrections to GR, could survive at low energies, but this is not impossible, in case, for example, when there is a low energy scale present in the matter Lagrangian. All in all, even in case the renormalizability conjecture fails, the class of theories (2) is of interest as a very simple class of “modified gravity” theories that does not contain new degrees of freedom.

4. Conclusions

A class of gravitational theories propagating two degrees of freedom and formulated without any direct reference to the spacetime metric was described. A theory from this class is specified by an (arbitrary) function ϕ of two arguments. General relativity belongs to the class considered and corresponds to $\phi = \text{const} = \Lambda$, Λ being the cosmological constant. One way to describe the nature of the modification is

to say that the cosmological constant became a non-trivial function of the “curvature”. However, the theories we consider cannot be obtained by simply inserting a scalar function of the Weyl curvature into the Lagrangian. The construction is much more subtle, and involves, in particular, replacing the paradigm of gravity being a dynamical theory of the metric by a new paradigm of gravity being about the dynamics of the Hodge dual operator. We have described some simple physical consequences of this modification of gravity, as well as reviewed the status of the question of coupling these theories to matter. Thus, gauge fields couple seamlessly, but no descriptions for fermions in this framework is as of yet known. Work is currently in progress on this very important issue.

It is tempting to try to apply the new class of gravity theories to the fundamental problems of cosmology, namely those of “dark matter” and “dark energy”. We have indicated how this might be possible, but much more work is needed, in particular on coupling of these theories to massive matter fields, to see if any realistic model can be built along these lines.

We have reviewed a conjecture to the effect that the described class of theories (with varying ϕ) is closed under the renormalization. Some arguments in support of this conjecture were given, but its status remains open.

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